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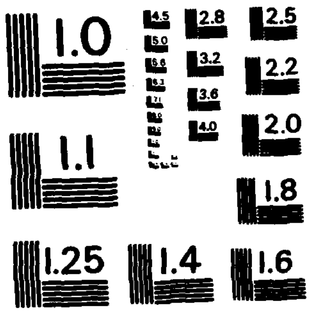

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A FREE BOUNDARY ARISING FROM McKEAN'S MODEL FOR NERVE CONDUCTION

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ABSTRACT

Considered is the pure initial value problem for the system of equations $u_t = u_{xx} + f(u) - w$, $w_t = \varepsilon(u-\gamma w)$, the initial data being $(u(x,0),w(x,0)) = (\phi(x),0)$. Here ε, γ are positive constants, and $f(u) = -u+H(u-a)$ where H is the Heaviside step function and $a \in (0,1/2)$. This system is of the FitzHugh-Nagumo type and models the conduction of electrical impulses in a nerve axon. In an earlier paper the author considered the curve $s(t) = \sup\{x:u(x,t) = a\}$, and showed that if $\phi(x) > a$ on a sufficiently long interval and decays sufficiently fast to zero as $|x| \to \infty$, then $\lim_{t\to\infty} s(t) = \infty$. In this paper a more detailed description of the asymptotic behavior of $s(t)$ is given. The results demonstrate that when ε is small, $s(t)$ eventually propagates at a rate close to the speed of the unique traveling wave solution, $U(z)$, for $\varepsilon = 0$, $w(x,t) \equiv 0$ which satisfies $U(-\infty) = 1$, $U(+\infty) = 0$.

AMS (MOS) Subject Classification: 35K55

Key Words: *Free boundary, Traveling Wave Solution*

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SIGNIFICANCE AND EXPLANATION

A model for the conduction of electrical impulses in a nerve axon is considered. In an earlier paper the author demonstrated that the model exhibits a threshold phenomenon. This corresponds to the biological fact that a minimum stimulus is required to trigger a nerve impulse. In this paper a more detailed description of the asymptotic behavior of the solution of the equations is given. It is proven that, in some sense, the solution eventually propagates with constant velocity.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A FREE BOUNDARY ARISING FROM McKEAN'S MODEL FOR NERVE CONDUCTION

David Terman

Section 1. Introduction

In this paper we consider the initial value problem for the system:

$$(1.1)\qquad \begin{cases} u_t = u_{xx} + f(u) - w \\ w_t = \varepsilon(u - \gamma w) \quad , \end{cases}$$

the initial conditions being $u(x,0) = u_0(x)$, $w(x,0) = 0$. It is assumed that ε and γ are positive constants and $f(u) = -u+H(u-a)$. Here H is the Heaviside step function and $a \in (0,1/2)$. This system, with $f(u)$ replaced by $f_1(u) = u(1-u)(u-a)$, was introduced by FitzHugh [9] and Nagumo, Arimoto and Yoshizawa [14] as a model for the condution of electrical impulses in a nerve axon. The model we consider was suggested by McKean [12].

In [18] it was demonstrated that (1.1) exhibits a threshold phenomenon. That is, if $u_0(x)$, which corresponds to the initial stimulus, is sufficiently small, then $\lim_{t\to\infty} \|u(\cdot,t)\|_{L^\infty} + \|w(\cdot,t)\|_{L^\infty} = 0$. In this case the initial data is said to be subthreshold. If $u_0(x)$ is sufficiently large, or superthreshold, then one expects some sort of signal to propagate. This was shown to be the case for ε sufficiently small. More precisely, assume that $u_0(x)$ satisfies the conditions:

(a) $u_0(x) \in C^2(\mathbb{R})$,

(b) $u_0(x) \in [0,1]$ in $\mathbb{R}$,

(c) $u_0(x) = u_0(-x)$,

(1.2) (d) there exists a unique constant $x_0 > 0$ such that $u_0(x_0) = a$,

(e) $u_0(x) > a$ for $|x| < x_0$,

(f) $|u_0(x)| < ae^{(\sqrt{2}/2)(x_0-x)}$ for $|x| > x_0$.

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Let $s(t)$ be defined as $s(t) = \sup\{t : u(x,t) = a\}$. We define $u_0(x)$ to be superthreshold if $\lim_{t\to\infty} s(t) = \infty$. The main result in [18] is then

<u>Theorem 1</u>: Choose $a \in (0,1/2)$ and $\gamma > 0$. Then there exist positive constants ε_0 and θ such that if $\varepsilon \in (0,\varepsilon_0)$ and $u_0(x)$ satisfies (1.2) with $x_0 > \theta$, then $u_0(x)$ is superthreshold.

In this paper we give a more detailed description of the asymptotic behavior of $s(t)$. Before stating these results we point out that one expects that if the initial data is superthreshold, then the solution of (1.1) should asymptotically approach a traveling wave solution. By a traveling wave solution we mean a solution of the form $(u(x,t), w(x,t)) = (U(z),W(z))$, $z = x + \theta t$. These correspond to solutions which propagate with constant shape and velocity. The existence of traveling wave solutions for the FitzHugh-Nagumo system, with $f_1(v)$, was given by Carpenter [2], Conley [3], and Hastings [10]. Rinzel and Keller [16] considered the McKean model with $\gamma = 0$. Similar results for the McKean model with $\gamma > 0$ have been obtained by Rinzel and Terman [17].

There may exist at least two traveling waves of a particular type (pulses, fronts, periodic waves, etc.). Jones [11] considered pulse shaped solutions of the FitzHugh-Nagumo model and showed that the fastest wave is asymptotically stable. This means that if the initial data is sufficiently close to the traveling wave, then the solution asymptotically approaches some translate of the wave. Stability of waves for McKean's model was proven by Feroe [8]. Both Jones and Feroe used techniques developed by Evans [4] - [7]. One expects, however, more to be true. Numerical calculations (see [17]) indicate that any superthreshold initial data should give rise to a solution which asymptotically approaches a translate of a traveling wave solution. In terms of the curve $s(t)$, this means that there should exist a constant, θ_ε, such that if ε is sufficiently small and the initial data is superthreshold, then $\lim_{t\to\infty} (s(t)-\theta_\varepsilon t)$ exists.

If $\varepsilon = 0$ and $w(x,t) \equiv 0$, then (1.1) reduces to the scalar equation:

(1.3) $$u_t = u_{xx} + f(u) \quad .$$

This equation possesses a unique traveling wave which satisfies $U(-\infty) = 1$, $U(+\infty) = 0$. In fact, if θ^* is the speed of this wave, then $\theta^* = (1-2a)[a(1-a)]^{-1/2}$ (see Rinzel and Keller [16]). The result of Jones and numerical calculations lead one to expect that $\lim_{\varepsilon \to 0} \theta_\varepsilon = \theta^*$. In this paper we show that if ε is sufficiently small, then $s(t)$ asymptotically propagates with speed close to θ^*. This is made precise in the following theorem. In the theorem, as in the rest of the paper, we assume that a and γ are fixed constants and $u_0(x)$ satisfies (1.2) with $x_0 > \Theta$.

Theorem 2: Given $\delta > 0$, there exists a constant, $\varepsilon(\delta)$, such that if $0 < \varepsilon < \varepsilon(\delta)$, then for any $T > 0$ there exists a $t_0 > T$ such that

$$(\theta^*-\delta)(t-t_0) + s(t_0) - \delta < s(t) < (\theta^*+\delta)(t-t_0) + \delta$$

for all $t > t_0$.

The theorem is proved using the fact that $s(t)$ must satisfy an integral equation. This equation is presented in the next section where preliminary results are also proved. In section 3 we describe how to choose $\varepsilon(\delta)$. The proof of Theorem 2 is given in section 4.

Throughout this paper we assume that there exists a unique solution of (1.1). We also assume that there exist constants U and W such that $|u(x,t)| < U$ and $|w(x,t)| < W$ in $\mathbb{R} \times \mathbb{R}^+$. The question of uniqueness is apparently quite hard and to the author's knowledge has not been solved. Some discussion of these questions is presented in [18].

We conclude the introduction by pointing out that in [19], the author considered the scalar equation (1.3) and showed that in the superthreshold case $\lim_{t\to\infty} (s(t)-\theta^* t)$ exists. A rather complete description of the asymptotic behavior of solutions of the scalar equation has been given by McKean [13]. McKean's results also hold for (1.1) with $\varepsilon < 0$.

Section 2. Preliminary Results

We first introduce some notation. Let

$$K(x,t) = \frac{e^{-t}}{2\pi^{1/2}t^{1/2}} e^{-x^2/4t} \quad . \tag{2.1}$$

That is, $K(x,t)$ is the fundamental solution of the equation

$$\psi_t = \psi_{xx} - \psi \quad . \tag{2.2}$$

It will be convenient to define $s(t)$ for $t < 0$. We assume that $s(t) = s(0)$ for $t < 0$. Let

$$\hat{\psi}(x,t) = \int_{-\infty}^{\infty} K(x-\xi,t)u_0(\xi)d\xi \quad ,$$

$$\hat{\phi}(x,t) = \int_{-\infty}^{t} \int_{-\infty}^{s(\tau)} K(x-\xi,t-\tau)d\xi d\tau \quad ,$$

$$\hat{\Gamma}(x,t) = \int_0^t \int_{-\infty}^{\infty} K(x-\xi,t-\tau)w(\xi,\tau)d\xi d\tau \quad ,$$

$$G = \{(x,t) : u(x,t) > a\} \quad ,$$

$$H = \{(x,t) : u(x,t) = a\} \quad .$$

Let $X(x,t)$ be the indicator function of the set G. Then for $(x,t) \notin H$, (u,w) satisfies the system

$$\begin{aligned} u_t &= u_{xx} - u + X(x,t) - w \\ w_t &= \varepsilon(u-\gamma w) \quad . \end{aligned} \tag{2.3}$$

If $(x,t) \notin H$, then (u,w) can be written implicitly as

$$\begin{aligned} u(x,t) &= \hat{\psi}(x,t) + \int_0^t \int_{-\infty}^{\infty} K(x-\xi,t-\tau)X(\xi,\tau)d\xi d\tau - \hat{\Gamma}(x,t) \\ w(x,t) &= \varepsilon e^{-\varepsilon\gamma t} \int_0^t e^{\varepsilon\gamma\eta}u(x,\eta)d\eta \quad . \end{aligned} \tag{2.4}$$

Letting $x = s(t)$ in (2.4) we obtain

$$\begin{aligned} &\text{(a)} \quad a = \hat{\psi}(s(t),t) + \int_0^t \int_{-\infty}^{\infty} K(s(t)-\xi,t-\tau)X(\xi,\tau)d\xi d\tau - \hat{\Gamma}(s(t),t) \\ &\text{(b)} \quad w(s(t),t) = \varepsilon e^{-\varepsilon\gamma t} \int_0^t e^{\varepsilon\gamma\eta}u(s(t),\eta)d\eta \quad . \end{aligned} \tag{2.5}$$

We now set

$$\Phi(s)(t) = \hat{\Phi}(s(t),t)$$

$$\Gamma(t) = \hat{\Gamma}(s(t),t)$$

$$\psi(t) = \hat{\psi}(s(t),t)$$

$$R(t) = \Phi(s)(t) - \int_0^t \int_{-\infty}^{\infty} K(s(t)-\xi,t-\tau)X(\xi,\tau)d\xi d\tau \quad .$$

Then (2.5) becomes

$$\text{(2.6)} \qquad \begin{aligned} &\text{(a)} \qquad \Phi(s)(t) = a + \Gamma(t) - \psi(t) + R(t) \\ &\text{(b)} \qquad w(s(t),t) = \varepsilon e^{-\varepsilon\gamma t} \int_0^t e^{\varepsilon\gamma\eta} u(s(t),\eta)d\eta \quad . \end{aligned}$$

We now present the important properties of the various terms appearing in (2.6), beginning with $\psi(t)$. Note that $\hat{\psi}(x,t)$ is the solution of the scalar equation (2.2) with initial condition $\psi(x,0) = u_0(x)$. A simple application of the usual comparison theorem for scalar, parabolic equations (see [1], Theorem 2.1) implies that $|\hat{\psi}(x,t)| < e^{-t}$ for all (x,t). Hence

$$\text{(2.7)} \qquad 0 < \psi(t) < e^{-t}$$

for all $t \in R^+$.

To estimate $R(t)$ we use the following result which is proved in [18].

<u>Proposition 2.1</u>: *Fix* a *and* γ, *and let* ε_0 *and* Θ *be as in Theorem 1.* Assume that $0 < \varepsilon < \varepsilon_0$ and $u_0(x)$ satisfies (1.2) with $x_0 > \Theta$. Then, there exists a positive constant λ $(=\lambda(\varepsilon))$ such that $u(x,t) > a$ for $s(t) - \lambda < x < s(t)$, $t > 0$. Furthermore, $\lim_{\varepsilon \to 0} \lambda(\varepsilon) = +\infty$.

This result implies that

$$\begin{aligned} R(t) &< \int_{-\infty}^0 \int_{-\infty}^{s(0)} K(s(t)-\xi,t-\tau)d\xi d\tau + \int_0^t \int_{-\infty}^{s(t)-\lambda(\varepsilon)} K(s(t)-\xi,t-\tau)d\xi d\tau \\ &\equiv R_1(t) + R_2(t) \quad . \end{aligned}$$

Now,

$$R_1(t) < \int_{-\infty}^0 \int_{-\infty}^{\infty} K(s(t)-\xi,t-\tau)d\xi d\tau = \int_{-\infty}^0 e^{-(t-\tau)} d\tau = e^{-t} \quad .$$

Since $\lim_{\varepsilon \to 0} \lambda(\varepsilon) = \infty$, a straightforward calculation shows that

$$R_2(t) < g(\varepsilon)$$

for some function, $g(\varepsilon)$, which satisfies $\lim_{\varepsilon \to 0} g(\varepsilon) = 0$.

Therefore,

(2.8) $$R(t) < g(\varepsilon) + e^{-t} \quad .$$

We now discuss some properties of Φ. Choose $T > 0$, and let $\alpha(t)$ and $\beta(\tau)$ be continuous functions which satisfy $\alpha(t) < \beta(t)$ for $t < T$ and $\alpha(T) = \beta(T)$. From the definition of Φ, it is clear that $\Phi(\alpha)(T) < \Phi(\beta)(T)$. Now let θ and K be constants and $\ell(t) = \theta t + K$. We claim that $\Phi(\ell)(t)$ is independent of both K and t. This is because,

$$\Phi(\ell)(t) = \int_{-\infty}^{t} \int_{-\infty}^{\theta\tau+K} K(\theta t+K-\xi, t-\tau)\, d\xi\, d\tau \quad ,$$

and, letting $s = \tau - t$, $z = \xi - (\theta t+K)$ we find that

$$\Phi(\ell)(t) = \int_{-\infty}^{0} \int_{-\infty}^{\theta s} K(-z,-s)\, dz\, ds \quad ,$$

which only depends on θ. Let

(2.9) $$h(\theta) = \int_{-\infty}^{0} \int_{-\infty}^{\theta s} K(-z,-s)\, dz\, ds \quad .$$

It is not hard to see that $h(0) = 1/2$, $h'(\theta) < 0$ for all θ, and $\lim_{\theta \to \infty} h(\theta) = 0$. Hence, there exists a unique constant θ^* such that $h(\theta^*) = a$. As discussed in [19], θ^* is, as in section 1, the speed of the unique traveling wave solution of (1.3) which satisfies $U(-\infty) = 1$ and $U(+\infty) = 0$.

Before discussing $\Gamma(t)$ we must discuss ways of finding a-priori bounds on $|u(x,t)|$ and $|w(x,t)|$. The main tool in obtaining these bounds is the next proposition and its corollaries. For the statement of these results it is convenient to define the operators:

$$F(u,w) = u_t - u_{xx} + u + w$$

$$G(u,w) = w_t - \varepsilon(u - \gamma w) \quad .$$

We fix constants c, K, and T with c and T positive. Let $\ell(t) = ct + K$ and $\Omega = \{(x,t) : x > \ell(t),\ 0 < t < T\}$. We assume that (u,w) is only defined in Ω, and satisfies (1.1) along with the initial-boundary conditions $(u(x,0),w(x,0)) = (u_0(x),w_0(x))$ for $x > K$, and $u(\ell(t),t) = \alpha(t)$ for $t \in [0,T]$. It is assumed that $\alpha(t)$, $u_0(x)$, and $w_0(x)$ are continuously differentiable. Assume that there exist functions $\underline{u}(x,t)$, $\bar{u}(x,t)$, $\underline{w}(x,t)$ and $\bar{w}(x,t)$ defined in Ω which satisfy

(2.10)

(a) $(\underline{u},\underline{w}) \leq (\bar{u},\bar{w})$ in Ω ,

(b) $\underline{u}$, $\bar{u}$, $\underline{w}$, and $\bar{w}$ along with their first derivatives with respect to t and second derivatives with respect to x are continuous functions in Ω ,

(c) $(\underline{u}(x,0),\underline{w}(x,0)) \leq (u_0(x),w_0(x)) \leq (\bar{u}(x,0),\bar{w}(x,0))$ for $x > K$,

(d) $\underline{u}(\ell(t),t) \leq \alpha(t) \leq \bar{u}(\ell(t),t)$ for $t \in [0,T]$.

Here $(a,b) \leq (c,d)$ means that $a \leq c$ and $b \leq d$.

<u>Proposition 2.2</u>: If (2.10) is satisfied and

$$(F(\underline{u},\bar{w}),G(\underline{u},\underline{w})) \leq (0,0) \leq (F(\bar{u},\underline{w}),G(\bar{u},\bar{w}))$$

in Ω, than $(\underline{u},\underline{w}) \leq (u,w) \leq (\bar{u},\bar{w})$ in Ω.

This result is proved in [20]. The following few results show how the proposition is used to obtain bounds on $w(x,t)$ and $\Gamma(t)$.

<u>Corollary 2.3</u>: Let $W = \dfrac{\varepsilon a}{c^2+\varepsilon\gamma}$ and assume that $\varepsilon < c^2$. Furthermore, assume that $|\alpha(t)| < a$ in $(0,T)$, $|u_0(x)| < ae^{c(K-x)}$ and $|w_0(x)| < We^{c(K-x)}$ for $x > K$. Then $|u(x,t)| \leq ae^{c(\ell(t)-x)}$ and $|w(x,t)| \leq We^{c(\ell(t)-x)}$ in Ω.

<u>Proof</u>: This result follows immediately from the proposition once we set $\underline{u}(x,t) = -ae^{c(\ell(t)-x)}$, $\bar{u}(x,t) = ae^{c(\ell(t)-x)}$, $\underline{w}(x,t) = -We^{c(\ell(t)-x)}$, and $\bar{w}(x,t) = We^{c(\ell(t)-x)}$.

<u>Proposition 2.4</u>: Assume that $u_0(x)$ satisfies (1.2) with $x_0 > \Theta$, and $0 < \varepsilon < \varepsilon_0$. There exists a constant M_1, such that $|w(s(t),t)| < \varepsilon M_1$ for all $t \in \mathbb{R}^+$.

Proof: In [18] it is proved that there exist positive constants λ_0, C_1, and C_2, which do not depend on ε, and a sequence $\{t_k\}$, $k = 0,1,2,\ldots$, such that for each k,

$$
(2.11)\qquad
\begin{array}{lll}
\text{(a)} & 0 = t_0 < t_1 < t_2 < \cdots & \\
\text{(b)} & C_1 < t_{k+1} - t_k < C_2 & \\
\text{(c)} & s(t_k) = x_0 + k & \\
\text{(d)} & s(t) < x_0 + k & \text{for } t < t_k \\
\text{(e)} & s(t) > x_0 + k - \lambda_0 & \text{for } t_k < t < t_{k+1} \\
\text{(f)} & |w(x,t)| < a & \text{for } x > s(t) \ .
\end{array}
$$

We first show that $|w(s(t_k),t_k)| < \varepsilon M_2$ for some M_2 and each k. Note that (2.11) implies that $s(t) < \frac{1}{C_2}(t - (t_k - C_2)) + s(t_k) \equiv \ell_k(t)$ for $t < t_k - C_2$. Fix k and let $\ell(t) = \ell_k(t)$. From the definition of $s(t)$ it follows that $u(\ell(t),t) < a$ for $t < t_k$. We claim that $u(\ell(t),t) > -a$ for $t < t_k$. This follows from the following comparison argument which shows that $u(x,t) > -a$ for $x > s(t)$.

Let L be the operator defined by $Lu \equiv u_t - u_{xx} + u$. If we can show that $Lu > L(-a)$ for $x > s(t)$ then, because $u(x,0) > -a$ for $x > s(0)$ and $u(s(t),t) > -a$, the usual comparison theorem for parabolic equations will imply that $u(x,t) > -a$ for $x > s(t)$. However, $L(-a) = -a$ while (2.11f) implies that $Lu = -w > -a$.

We assume that $C_2 > \sqrt{2}$. Then, using (1.2f), $|u(x,0)| < ae^{1/C_2(x_0-x)}$ for $x > x_0$. Corollary 2.3 now implies that $|w(\ell(t),t)| < \dfrac{a\varepsilon}{(1/C_2)^2+\varepsilon\gamma} < aC_2^2\varepsilon$ for $t < t_{k-1}$. In particular, $|w(s(t_k),t_k-C_2)| < aC_2^2\varepsilon$. Then, for $t > t_k - C_2$, the equation $w_t = \varepsilon(u-\gamma w)$ implies that

$$|w(s(t_k),t)| \le |w(s(t_k),t_k-C_2)| + \varepsilon \int_{t_k-C_2}^{t} |u(s(t_k),\eta)|d\eta \ .$$

Therefore, letting $t = t_k$ and using the assumption that $|u(x,t)| < U$ in $\mathbb{R}\times\mathbb{R}^+$, we find that

$$|w(s(t_k),t_k)| \le aC_2^2\varepsilon + \varepsilon UC_2 \ .$$

So if we set $M_2 = aC_2^2 + UC_2$ we find that $|w(s(t_k),t_k)| < \varepsilon M_2$ for each k.

We must now estimate $w(s(t),t)$ for t not equal to one of the t_k's. Choose k so that $t_k < t < t_{k+1}$. We consider two cases. First assume that $t_k < (\lambda_0+1)C_2$. Then

$$|w(s(t),t)| < \varepsilon Ut < \varepsilon Ut_{k+1} < \varepsilon U(t_k+C_2) < \varepsilon UC_2(\lambda_0+2) \quad .$$

Next assume that $t_k > (\lambda_0+1)C_2$. Let $\ell^{-1}(x)$ be the inverse function of $\ell_k(t)$. That is,

$$\ell^{-1}(x) = C_2(x-s(t_k)) + t_k-C_2 \quad .$$

Now, (2.11e) implies that $s(t) > s(t_k) - \lambda_0$. Hence, $\ell^{-1}(s(t)) > t_k - (\lambda+1)C_2$. Hence,

$$|w(s(t),t)| < |w(s(t),\ell^{-1}(s(t)))| + \varepsilon \int_{\ell^{-1}(s(t))}^{t} |u(s(t),\eta)|d\eta$$

$$< aC_2^2\varepsilon + \varepsilon U[t_{k+1}-\ell^{-1}(s(t))]$$

$$< \varepsilon[aC_2^2 + U(\lambda_0+2)C_2] \quad .$$

So, we let $M_1 = aC_2^2 + U(\lambda_0+2)C_2$, and the proof of the proposition is complete.

<u>Proposition 2.5</u>: There exists a constant M, which does not depend on ε, such that $\Gamma(t) < \varepsilon M$ for all t.

<u>Proof</u>: Fix $t_0 > 0$, and let C_1 and C_2 be as in the Proposition 2.4. Let $\ell(t) = \frac{1}{C_2}[t-t_0] + s(t_0)$. The proof of Proposition 2.4 shows that $|w(x,t)| < M_1\varepsilon$ for $x > \ell(t)$, $0 < t < t_0$. Let $\ell^{-1}(x)$ be the inverse function of $\ell(t)$, and assume that $\ell(0) < x < \ell(t)$, $\ell^{-1}(x) < t < t_0$. Then,

$$|w(x,t)| < |w(x,\ell^{-1}(x))| + \varepsilon \int_{\ell^{-1}(x)}^{t} |u(x,\eta)|d\eta$$

$$< \varepsilon M_1 + \varepsilon U[t-\ell^{-1}(x)] < \varepsilon[M_1 + UC_2(x-s(t_0))] \quad .$$

A similar computation shows that if $x < \ell(0)$, then

$$|w(x,t)| < \varepsilon[M_1 + UC_2(x-s(t_0))] \quad .$$

Now,

$$|\Gamma(t_0)| < \int_0^{t_0} \int_{\ell(\tau)}^{\infty} K(s(t_0)-\xi,t_0-\tau)|w(\xi,\tau)|d\xi d\tau + \int_0^{t_0} \int_{-\infty}^{\ell(\tau)} K(s(t_0)-\xi,t_0-\tau)|w(\xi,\tau)|d\xi d\tau$$

$$= \text{(I)} + \text{(II)} \quad .$$

Note that,

$$\text{I} < \varepsilon M_1 \int_0^{t_0} \int_{-\infty}^{\infty} K(s(t_0)-\xi, t_0-\tau)\, d\xi\, d\tau < \varepsilon M_1 .$$

On the other hand,

$$\text{II} < \varepsilon \int_0^{t_0} \int_{-\infty}^{s(t_0)} K(s(t_0)-\xi, t_0-\tau)[M_1 + UC_2(\xi - s(t_0))]\, d\xi\, d\tau$$

$$< \varepsilon \int_{-\infty}^{t_0} \int_{-\infty}^{s(t_0)} K(s(t_0)-\xi, t_0-\tau)[M_1 + UC_2(\xi - s(t_0))]\, d\xi\, d\tau$$

$$= \varepsilon \int_{-\infty}^{0} \int_{-\infty}^{0} K(-\xi, -\tau)[M_1 - UC_2 \xi]\, d\xi\, d\tau$$

$$< \varepsilon M_2$$

for some constant M_2 which does not depend on t_0 or ε. Setting $M = M_1 + M_2$ we obtain the desired result.

Combining (2.6a), (2.7), (2.8), and Proposition 2.5 we conclude that there exists a constant ε_1 and a function $G(\varepsilon)$ with the property that $\lim_{\varepsilon \to 0} G(\varepsilon) = 0$ such that if $0 < \varepsilon < \varepsilon_1$, then

(2.12) $$|\Phi(s)(t) - a| < G(\varepsilon) + 2e^{-t} .$$

Section 3. The constant $\varepsilon(\delta)$.

Throughout the rest of the paper we assume that the constant δ, which appears in the statement of Theorem 2, is fixed. Later it will be convenient to assume that $\delta < \inf\{1, c^*/2\}$. Since we're really interested when δ is small, this is no problem. In this section we explain how to choose $\varepsilon(\delta)$. We begin with a few preliminary results.

Fix $\sigma > 0$, and $T > 0$. Let

$$\ell(t) = \sigma(t-T) + s(T) \qquad \text{for } t \in \mathbb{R}^+ ,$$

$$H = \{t < T : s(t) < \ell(t)\} ,$$

$$D = \{(x,t) : s(t) < x < \ell(t),\ t \in H\} ,$$

$$A(x,t) = \int_D\int K(x-\xi, t-\tau)\,d\xi\,d\tau \quad \text{for } t > T .$$

Proposition 3.1. Given $\alpha > 0$, and $x \in \mathbb{R}$, there exists β $(=\beta(\alpha))$ such that if $A(x,T) > \alpha$, then $A(s(T),T) > \beta$. Furthermore, β does not depend on x, T, σ, or ε.

Proof: Since we're really only interested when α is small, we assume that $\alpha < 1/3$. Let $M = -\log \alpha/3$, $M_2 = -\log(1 - \alpha/3)$, and $z(t) = C_2(t-T) + s(T) - \lambda_0$. Here C_2 and λ_0 are as in Proposition 2.4. That proposition implies that

$$z(t) < s(t) < s(T) + 2\lambda_0 \quad \text{for } t < T . \tag{3.1}$$

Let

$$\mathcal{M} = \{(\xi,\tau) : T-M_1 < \tau < T-M_2,\ z(\tau) < \xi < s(T) + 2\lambda_0\} ,$$

$$D_1 = \{(\xi,\tau) \in D : \tau < T-M_1\} ,$$

$$D_2 = \{(\xi,\tau) \in D : T-M_1 < \tau < T-M_2\} ,$$

$$D_3 = \{(\xi,\tau) \in D : T-M_2 < \tau < T\} .$$

Then, for $x \in \mathbb{R}$,

$$A(x,T) = \int_{D_1}\int K(x-\xi,T-\tau)\,d\xi\,d\tau + \int_{D_2}\int K(x-\xi,T-\tau)\,d\xi\,d\tau$$

$$+ \int_{D_3}\int K(x-\xi,T-\tau)\,d\xi\,d\tau$$

$$< \int_{-\infty}^{T-M_1}\int_{-\infty}^{\infty} K(x-\xi,T-\tau)\,d\xi\,d\tau + \int_{D_2}\int K(x-\xi,T-\tau)\,d\xi\,d\tau$$

$$+ \int_{T-M_2}^{T} \int_{-\infty}^{\infty} K(x-\xi,T-\tau)\,d\xi\,d\tau$$

$$< \frac{2\alpha}{3} + \int_{D_2}\int K(x-\xi,T-\tau)\,d\xi\,d\tau \quad .$$

If $A(x,T) > \alpha$, then

$$\int_{D_2}\int K(x-\xi,T-\tau)\,d\xi\,d\tau > \frac{\alpha}{3} \quad .$$

Note that if $|x|$ is sufficiently large, then for $(\xi,\tau) \in M$,

$$K(s(T)-\xi,T-\tau) > K(x-\xi,T-\tau) \quad .$$

Therefore,

$$r = \inf \frac{K(x-\xi,T-\tau)}{K(s(T)-\xi,T-\tau)} \quad \text{over} \quad (\xi,\tau) \in M,\ x \in R$$

is positive. Note that r does not depend on x, T, σ, or ε. Now (3.1) implies that $D_2 \subset M$. Hence,

$$\begin{aligned} A(s(T),T) &> \int_{D_2}\int K(s(T)-\xi,T-\tau)\,d\xi\,d\tau \\ &> r\int_{D_2}\int K(x-\xi,T-\tau)\,d\xi\,d\tau \\ &> \frac{r}{3}\,\alpha \quad . \end{aligned}$$

<u>Corollary 3.2</u>. Given $\alpha > 0$, let β be as in the proposition. If $A(s(T),T) < \beta$, then $A(x,t) < \alpha e^{T-t}$ for all $x \in R$, $t > T$.

<u>Proof</u>: Proposition 3.1 and the assumption that $A(s(T),T) < \beta$ implies that $A(x,T) < \alpha$ for all $x \in R$. Note that $A(x,t)$ is the solution of the inhomogeneous equation

$$\begin{aligned} \psi_t &= \psi_{xx} - \psi + \chi_D \quad && \text{in} \quad R \times R^+ \\ \psi(x,0) &= 0 && \text{in} \quad R \quad . \end{aligned}$$

Here, χ_D is the indicator function of the set D. "Restarting" at $t = T$ we find that $A(x,t)$ is the solution of the homogeneous equation

$$\begin{aligned} \psi_t &= \psi_{xx} - \psi \quad && \text{in} \quad R \times (T,\infty) \\ \psi(x,0) &= A(x,T) && \text{in} \quad R \quad . \end{aligned} \tag{3.2}$$

The result now follows from a simple application of the standard comparison theorem for parabolic equations.

We now describe, without motivation, how to choose $\varepsilon(\delta)$. Let

$$I = \{(x,c) : 0 \le x \le 1,\ c^*/2 \le c \le 2c^*\} ,$$

$$(3.3)\qquad M_1 = \inf_{(x,c)\in I} \int_{-\infty}^{-1} F(x-c\tau,-\tau)d\tau ,$$

$$\alpha = \frac{M_1\,\delta}{2} ,$$

$$\beta = \beta(\alpha) .$$

Let $h(\theta)$ and $G(\varepsilon)$ be as in (2.9) and (2.12), respectively. It is assumed that $\varepsilon(\delta)$ is chosen so that if $0 \le \varepsilon \le \varepsilon(\delta)$ and $|h(\theta)-a| < 2G(\varepsilon)$, then

$$(3.4)\qquad |\theta-\theta^*| < \inf\{\delta/2,\ \beta(\alpha)/24\} .$$

We also assume that $G(\varepsilon) < \frac{1-2a}{4}$, and $\varepsilon(\delta) < \varepsilon_1$.

Section 4. Proof of Theorem 2

It is assumed throughout this section that $u_0(x)$ satisfies (1.2) with $x_0 > \theta$, and $0 < \varepsilon < \varepsilon(\delta)$. We begin by obtaining a lower bound on the rate at which $s(t)$ propagates.

Lemma 4.1: Let $c_1 = \sup\{c : s(t) > ct + K$ for some K and all $t\}$. Then $c_1 > \theta^* - \delta/2$.

Proof: Recall that $\varepsilon(\delta)$ was chosen so that if $0 < \varepsilon < \varepsilon(\delta)$ and $h(\theta) = a + G(\varepsilon)$, then $|\theta-\theta^*| < \delta/2$. Furthermore, $G(\varepsilon) < \frac{1-2a}{4}$. Choose N so that $\frac{a}{2N} < \frac{1-2a}{4}$. For $n > N$ choose c_n so that $h(c_n) = a + G(\varepsilon) + \frac{a}{2n}$. Let $t_n = -\log \frac{a}{4n}$ and $\lambda = \frac{1}{2} \inf\limits_{0<t<t_N} s(t)$. For $t \in [0,t_N]$ let $\alpha(t) = \lambda$, and for $t > t_N$, let $\alpha(t)$ be the continuous, piecewise linear function defined by $\alpha'(t) = c_n$ for $t \in (t_n,t_{n+1})$. We show that $\alpha(t) < s(t)$ in R^+. Since $\alpha'(t) > \theta^* - \delta$ for t sufficiently large, this will complete the proof.

It is certainly true that $\alpha(t) < s(t)$ for $t \in (0,T_N)$. Suppose that there exists $T > t_N$ such that $s(T) = \alpha(T)$ and $\alpha(t) < s(t)$ for $t < T$. Assume that $T \in [t_n,t_{n+1})$. Then, (2.12) implies that

$$\begin{aligned}\Phi(s)(T) &< a + G(\varepsilon) + 2e^{-T} \\ &< a + G(\varepsilon) + \frac{a}{2n} .\end{aligned}$$

On the other hand, if we let $\ell(t) = c_n(t-T) + s(T)$, then $s(t) > \alpha(t) > \ell(t)$ for $t < T$. Hence,

$$\Phi(s)(T) > \Phi(\alpha)(T) > \Phi(\ell)(T) = h(c_n) = a + G(\varepsilon) + \frac{a}{2n}$$

and we have a contradiction.

The proof of the following lemma is quite similar to the one just given.

Lemma 4.2. Let $c_0 = \inf\{c : s(t) < ct + K$ for some K and all $t\}$. Then $c_0 < \theta^* + \delta/2$.

Before preceeding it is necessary to introduce some notation. Choose σ_1, σ_2 so that $h(\sigma_1) = a - 2G(\varepsilon)$ and $h(\sigma_2) = a + 2G(\varepsilon)$. Since $\varepsilon < \varepsilon(\delta)$ it follows that $\sigma_1 - \sigma_2 < \delta$. Let $K_0 = \lim\limits_{t\to\infty} \sup\{s(t)-c_0t\}$, and $\ell_0(t) = c_0t + K_0$. Choose $\{t_n\}$ so that

$s(t) < \ell_0(t) + 1/n$ for $t > t_n$, and $s(t_n) > \ell_0(t_n) - 1/n$. Choose $\{\theta_n\}$ so that $h(\theta_n) = \Phi(s)(t_n)$, and let $\lambda_n(t) = \theta_n(t-t_n) + s(t_n)$. For the time being fix n. Assume that $2e^{-t_n} < G(\varepsilon)$. Let $T = t_n$, $\ell_1(t) = \sigma_1(t-T) + s(T)$, $\ell_2(t) = \sigma_2(t-T) + s(T)$, $\ell_3(t) = \lambda_n(t)$, and $c_3 = \theta_n$.

For $j = 0,1,2,$ and 3, let

$$H_j = \{t<T; s(t) < \ell_j(t)\}$$

$$J_j = \{t<T; \ell_j(t) < s(t)\}$$

$$A_j = \int_{H_j} \int_{s(\tau)}^{\ell_j(\tau)} K(s(T)-\xi, T-\tau)\,d\xi\,d\tau$$

$$B_j = \int_{J_j} \int_{\ell_j(\tau)}^{s(\tau)} K(s(T)-\xi, T-\tau)\,d\xi\,d\tau \quad .$$

To emphasize their dependence on n, we sometimes write $H_j(n)$, $J_j(n)$, $A_j(n)$, and $B_j(n)$. For $j = 0$ and 3, (2.12) implies that, $h(\sigma_1) < h(c_j) < h(\sigma_2)$, and, therefore, $\sigma_2 < c_j < \sigma_1$. From our choice of t_n we conclude that $\ell_0(t) < \ell_1(t) + 2/n$ for $t > T$. Furthermore, since $\ell_1(T) = \ell_3(T) = \ell_2(T) = s(T)$, it follows that $\ell_1(t) < \ell_3(t) < \ell_2(t)$ for $t < T$.

<u>Lemma 4.3</u>. $B_0(n) \to 0$ as $n \to \infty$.

<u>*Proof:*</u> Fix $m < n$. Since $s(t) < \ell_0(t) + \frac{2}{m}$ for $t > t_m$,

$$B_0(n) < \int_{-\infty}^{t_m} \int_{-\infty}^{\infty} K(s(t_n)-\xi, t_n-\tau)\,d\xi\,d\tau + \int_{t_m}^{t_n} \int_{\ell_0(\tau)}^{\ell_0(\tau)+\frac{2}{m}} K(s(t_n)-\xi, t_n-\tau)\,d\xi\,d\tau$$

$$= [I] + [II] \quad .$$

Now,

$$[I] < \int_{-\infty}^{t_m} e^{-(t_n-\tau)}\,d\tau < e^{-(t_n-t_m)} \quad .$$

On the other hand,

$$[II] < \int_{-\infty}^{t_n} \int_{\ell_0(\tau)}^{\ell_0(\tau)+\frac{4}{m}} K(s(t_n)-\xi, t_n-\tau)\,d\xi\,d\tau$$

$$= \int_0^{4/m} \int_{-\infty}^0 K(\eta - c^*\tau, -\tau)\,d\tau\,d\eta$$

$$< \frac{M}{m}$$

for some constant M which does not depend on m and n. We have now shown that

$$B_0(n) < \frac{M}{m} + e^{-(t_n - t_m)}$$

for all $m < n$. Let $m = \frac{n}{2}$ if n is even and $m = \frac{n+1}{2}$ if n is odd. It follows that $B_0(n) < \frac{2M}{n} + e^{-t_n/2}$, and the proof is complete.

Note that $A_3 = B_3$. This is because the equality $h(c_3) = \Phi(s)(T)$ implies that

$$\int_{-\infty}^T \int_{-\infty}^{\ell_3(\tau)} K(s(T)-\xi, T-\tau)\,d\xi\,d\tau = \int_{-\infty}^T \int_{-\infty}^{s(\tau)} K(s(T)-\xi, T-\tau)\,d\xi\,d\tau \quad .$$

Hence,

$$0 = \int_{-\infty}^T \int_{s(\tau)}^{\ell_3(\tau)} K(s(T)-\xi, T-\tau)\,d\xi\,d\tau = A_3 - B_3 \quad .$$

We now estimate $A_2(n)$. Note that since $\ell_1(t) < \ell_3(t) < \ell_2(t)$ for $t < T$, it follows that

$$\begin{aligned} A_2(n) &< A_3(n) + \int_0^T \int_{\ell_1(\tau)}^{\ell_2(\tau)} K(s(T)-\xi, T-\tau)\,d\xi\,d\tau \\ &= B_3(n) + \int_0^T \int_{\ell_1(\tau)}^{\ell_2(\tau)} K(s(T)-\xi, T-\tau)\,d\xi\,d\tau \\ &< B_0(n) + \left|\int_{-\infty}^T \int_{\ell_3(\tau)}^{\ell_0(\tau)} K(s(T)-\xi, T-\tau)\,d\xi\,d\tau\right| + \int_0^T \int_{\ell_1(\tau)}^{\ell_2(\tau)} K(s(T)-\xi, T-\tau)\,d\xi\,d\tau \\ &\equiv B_0(n) + [I] + [II] \quad . \end{aligned}$$

To estimate [I] we recall that for $j = 0$ or 3, $\sigma_2 < c_j < \sigma_1$. Moreover, $\ell_0(T) < s(T) + 1/n$. Hence, a simple calculation shows that

$$\begin{aligned} [I] &< \int_{-\infty}^T \int_{\ell_1(\tau)}^{\ell_2(\tau)+1/n} K(s(T)-\xi, T-\tau)\,d\xi\,d\tau = \int_{-\infty}^0 \int_{\sigma_1\tau}^{\sigma_2\tau} K(-\xi, -\tau)\,d\xi\,d\tau \\ &< 3(\sigma_2 - \sigma_1) \quad . \end{aligned}$$

Similarly,

$$[II] \le \int_{-\infty}^{0} \int_{\sigma_1\tau}^{\sigma_2\tau} K(-\xi,-\tau)d\xi d\tau$$

$$\le 3(\sigma_2-\sigma_1) \quad .$$

Therefore,

$$A_2(n) \le B_0(n) + 6(\sigma_2-\sigma_1) \quad .$$

We assume throughout that n is chosen so that $B_0(n) < \frac{\beta(\alpha)}{2}$ where $\beta(\alpha)$ was defined in (3.3). From (3.4) it follows that $\sigma_2-\sigma_1 < \frac{\beta(\alpha)}{12}$. Therefore, we conclude that for n sufficiently large,

$$A_2(n) < \beta(\alpha) \quad . \tag{4.1}$$

Here we give a brief outline of how the proof of Theorem 2 is completed. We construct a sequence of positive constants, $\{\delta_k\}$, $k = 0,1,2,\ldots$, such that $\sum_{k=0}^{\infty} \delta_k < \delta$. Furthermore, letting $z(t)$ equal to the piecewise continuous function defined by

$$z(t) = \begin{cases} s(t) & \text{for } t < t_n \\ \ell_2(t) - \sum_{j=0}^{k} \delta_j & \text{for } t_n + k \le t < t_n + k + 1 \quad , \end{cases} \tag{4.2}$$

for $k = 0,1,2,\ldots$, we show that, for n large, $z(t) < s(t)$ for $t > t_n$. This completes the proof of Theorem 2 for the following reason. Recall that $s(t) < \ell_1(t) + 2/n$ for $t > t_n$. Hence, if n is sufficiently large, then

$$\ell_2(t) - \delta < s(t) < \ell_1(t) + \delta$$

for $t > t_n$. However,

$$\ell_1'(t) - \ell_2'(t) = \sigma_1 - \sigma_2 < \delta \quad .$$

In what follows we fix n, let $T = t_n$, and $T_k = T + k$.

The δ_k are defined inductively. Assume that $\delta_1,\ldots,\delta_{k-1}$ have been chosen so that for $t < T_k$, $s(t) > z(t)$ where $z(t)$ is defined by (4.2). We show how to define δ_k. Here, k may be equal to zero. Note that δ_k must be chosen in such a way that $z(t) < s(t)$ for $t \in (T_k,T_{k+1})$. From the definition of δ_k it will be clear that $\sum_{k=0}^{\infty} \delta_k < \delta$ for n sufficiently large.

Assume, for the moment, that δ_k has been chosen, and there exists $t_0 \in (T_k, T_{k+1})$ such that $z(t_0) = s(t_0)$, and $z(t) < s(t)$ for $t \in (T_k, t_0)$. The following calculation demonstrates that if δ_k is too large, and n is suitable chosen, then

$$\Phi(z)(t_0) > h(\sigma_2) \quad .$$

Since $z(t) < s(t)$ for $t < t_0$ this implies that

$$\Phi(s)(t_0) > \Phi(z)(t_0) > h(\sigma_2) = a + 2G(\varepsilon)$$

which contradicts (2.12). In what follows we set

$$z_k(t) = \ell_2(t) - \sum_{j=0}^{k} \delta_j \quad .$$

Now,

$$\Phi(z)(t_0) = \int_{-\infty}^{t_0} \int_{-\infty}^{z_k(\tau)} K(z(t_0)-\xi, t_0-\tau)d\xi d\tau + \int_{-\infty}^{t_0} \int_{z_k(\tau)}^{z(\tau)} K(z(t_0)-\xi, t_0-\tau)d\xi d\tau$$

(4.3)

$$> h(\sigma_2) + \int_{-\infty}^{t_k} \int_{z_k(\tau)}^{z_{k-1}(\tau)} K(z(t_0)-\xi, t_0-\tau)d\xi d\tau + \int_{-\infty}^{t_0} \int_{z_{k-1}(\tau)}^{s(\tau)} K(z(t_0)-\xi, t_0-\tau)d\xi d\tau$$

$$\equiv h(\sigma_2) + [I] + [II] \quad .$$

Note that

$$[I] = \int_{-\infty}^{-1} \int_{\sigma_2\tau}^{\sigma_2\tau+\delta_k} K(-\xi, -\tau)d\xi d\tau$$

(4.4)

$$> \delta_k M_1$$

where M_1 was defined in the previous section. Here is where we used the assumption, mentioned in the beginning of section 3, that $\delta < \inf\{1, c^*/2\}$. To estimate [II] we note that, since $z_{k-1}(\tau) < \ell_2(\tau)$ for $\tau < t_0$,

$$\{\tau : s(\tau) < z_{k-1}(\tau) : \tau < t_0\} \subset H_2 \quad .$$

Hence,

(4.5) $$[II] > -\int_{H_2} \int_{s(\tau)}^{\ell_2(\tau)} K(z(t_0)-\xi, t_0-\tau)d\xi d\tau \quad .$$

We now wish to apply Corollary 3.2. Comparing the notation of sections 3 and 4 we find that if $\ell(t) = \ell_2(t)$, then

(4.6) $$A(x,t) = \int_{H_2} \int_{s(\tau)}^{\ell_2(\tau)} K(x-\xi, t-\tau)d\xi d\tau$$

for $t > T$, and

$$A(s(T),T) = A_2(n) \quad .$$

Hence (4.1) implies that if n is sufficiently large, then $A(s(T),T) < \beta(\alpha)$, and we conclude from Corollary 3.2, (4.5) and (4.6) that

$$\text{(4.7)} \qquad [II] > -\alpha e^{T-t_0} > -\alpha e^{-k} \quad .$$

Combining (4.3), (4.4), and (4.7) we find that

$$\Phi(z)(t_0) > h(\sigma_2) + \delta_k M_1 - \alpha e^{-k} \quad .$$

So if we set $\delta_k = \frac{\alpha}{M_1} e^{-k}$, we obtain the desired result that $\Phi(z)(t_0) > h(\sigma_2)$. It remains to verify that $\sum_{k=0}^{\infty} \delta_k < \delta$. However, this follows from our choice of α given in (3.3).

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Considered is the pure initial value problem for the system of equations $u_t = u_{xx} + f(u) - w$, $w_t = \varepsilon(u-\gamma w)$, the initial data being $(u(x,0),w(x,0)) = (\phi(x),0)$. Here ε, γ are positive constants, and $f(u) = -u+H(u-a)$ where H is the Heaviside step function and $a \in (0,1/2)$. This system is of the FitzHugh-Nagumo type and models the conduction of electrical impulses in a nerve axon. In an earlier paper the author considered the curve $s(t) = \sup\{x:u(x,t) = a\}$, and showed that if $\phi(x) > a$ on a sufficiently long interval and decays

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ABSTRACT (continued)

sufficiently fast to zero as $|x| \to \infty$, then $\lim_{t\to\infty} s(t) = \infty$. In this paper a more detailed description of the asymptotic behavior of $s(t)$ is given. The results demonstrate that when ε is small, $s(t)$ eventually propagates at a rate close to the speed of the unique traveling wave solution, $U(z)$, for $\varepsilon = 0$, $w(x,t) \equiv 0$ which satisfies $U(-\infty) = 1$, $U(+\infty) = 0$.

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